

worker's effort in period two implicitly as a function of \hat{a}_1 , σ_1^2 , and σ_ε^2 . Denote this function by α_2 . Note that

$$J_2(a, \hat{a}_1, \sigma_1^2, \sigma_\varepsilon^2) \geq g'(a) - 1 - a.$$

Hence, regardless of \hat{a}_1 , σ_1^2 , and σ_ε^2 , $\alpha_2(\hat{a}_1, \sigma_1^2, \sigma_\varepsilon^2)$ is bounded above by \bar{a}_2 , where $\bar{a}_2 > 0$ is the unique solution to $g'(a) = 1 + a$. In other words, α_2 is uniformly bounded. Since $g''(a) \geq g'(a)/a$ for all $a > 0$, we have from (17) that

$$\begin{aligned} \frac{\partial J_2}{\partial a}(\alpha_2(\hat{a}_1, \sigma_1^2, \sigma_\varepsilon^2), \hat{a}_1, \sigma_1^2, \sigma_\varepsilon^2) &= g''(\alpha_2(\hat{a}_1, \sigma_1^2, \sigma_\varepsilon^2)) - \frac{(1 + \hat{a}_1)\sigma_1^2}{[1 + (1 + \hat{a}_1)^2]\sigma_1^2 + \sigma_\varepsilon^2} \\ &\geq \frac{1 + \hat{a}_1 + \alpha_2(\hat{a}_1, \sigma_1^2, \sigma_\varepsilon^2)}{\alpha_2(\hat{a}_1, \sigma_1^2, \sigma_\varepsilon^2)} \frac{(1 + \hat{a}_1)\sigma_1^2}{\{[1 + (1 + \hat{a}_1)^2]\sigma_1^2 + \sigma_\varepsilon^2\}} - \frac{(1 + \hat{a}_1)\sigma_1^2}{[1 + (1 + \hat{a}_1)^2]\sigma_1^2 + \sigma_\varepsilon^2} > 0. \end{aligned}$$

Thus, α is differentiable in \hat{a}_1 , σ_1^2 , and σ_ε^2 by the Implicit Function Theorem.²¹

It is immediate to see from (18) that α_2 is strictly increasing in σ_1^2 and strictly decreasing in σ_ε^2 . The effect of an increase in \hat{a}_1 on α_2 is ambiguous, though. Indeed,

$$\begin{aligned} \frac{\partial \alpha_2}{\partial \hat{a}_1}(\hat{a}_1, \sigma_1^2, \sigma_\varepsilon^2) &= \frac{1}{g''(\alpha_2(\hat{a}_1, \sigma_1^2, \sigma_\varepsilon^2)) - \frac{(1 + \hat{a}_1)\sigma_1^2}{[1 + (1 + \hat{a}_1)^2]\sigma_1^2 + \sigma_\varepsilon^2}} \cdot \frac{\sigma_1^2}{[1 + (1 + \hat{a}_1)^2]\sigma_1^2 + \sigma_\varepsilon^2} \\ &\quad \times \left\{ 2(1 + \hat{a}_1) + \alpha_2(\hat{a}_1, \sigma_1^2, \sigma_\varepsilon^2) - \frac{2[1 + \hat{a}_1 + \alpha_2(\hat{a}_1, \sigma_1^2, \sigma_\varepsilon^2)](1 + \hat{a}_1)^2\sigma_1^2}{[1 + (1 + \hat{a}_1)^2]\sigma_1^2 + \sigma_\varepsilon^2} \right\}. \quad (19) \end{aligned}$$

Therefore, $\partial \alpha_2(\hat{a}_1, \sigma_1^2, \sigma_\varepsilon^2)/\partial \hat{a}_1 > 0$ when \hat{a}_1 is small. However, since α_2 is uniformly bounded, $\partial \alpha_2(\hat{a}_1, \sigma_1^2, \sigma_\varepsilon^2)/\partial \hat{a}_1 < 0$ when \hat{a}_1 is large. The intuition for the non monotonicity of α_2 in \hat{a}_1 is as follows. The worker's wage in any period is proportional to both his expected human capital and his reputation. Thus, an increase in \hat{a}_1 , which implies an increase in the worker's expected human capital in period three, increases the worker's return from manipulating his period three reputation. On the other hand, the effect of the worker's output in period two on his reputation in period three decreases as \hat{a}_1 increases, making it more costly for the worker to influence market beliefs in the third period. When \hat{a}_1 is small, the first effect dominates the second. When \hat{a}_1 is large, the second effect prevails.

Suppose now the worker's effort in period two is $\alpha_2(\hat{a}_1, \sigma_1^2, \sigma_\varepsilon^2)$. An argument similar to the one used in the main text shows that the worker's optimal choice of effort in the first

²¹We are implicitly assuming that for each σ_1^2 and σ_ε^2 , $\alpha_2(\cdot, \sigma_1^2, \sigma_\varepsilon^2)$ is defined in a neighborhood of zero. This is not a problem since (18) has a unique and positive solution when \hat{a}_1 is in a neighborhood of zero.

period is the solution to

$$\frac{(1 + \widehat{a}_1)\sigma_1^2}{\sigma_1^2 + \sigma_\varepsilon^2} + \frac{[1 + \widehat{a}_1 + \alpha_2(\widehat{a}_1, \sigma_1^2, \sigma_\varepsilon^2)]\sigma_1^2}{[1 + (1 + \widehat{a}_1)^2]\sigma_1^2 + \sigma_\varepsilon^2} + \frac{(1 + \widehat{a}_1)[1 + \widehat{a}_1 + \alpha_2(\widehat{a}_1, \sigma_1^2, \sigma_\varepsilon^2)]\sigma_1^2 m_1}{[1 + (1 + \widehat{a}_1)^2]\sigma_1^2 + \sigma_\varepsilon^2} - r \left\{ \frac{(1 + \widehat{a}_1)[1 + \widehat{a}_1 + \alpha_2(\widehat{a}_1, \sigma_1^2, \sigma_\varepsilon^2)]\sigma_1^2}{[1 + (1 + \widehat{a}_1)^2]\sigma_1^2 + \sigma_\varepsilon^2} \right\}^2 [1 + a + \alpha_2(\widehat{a}_1, \sigma_1^2, \sigma_\varepsilon^2)]\sigma_1^2 + \lambda = g'(a),$$

where $\lambda \geq 0$ and $\lambda = 0$ if the solution is positive. The term

$$MB(\widehat{a}_1, \sigma_1^2, \sigma_\varepsilon^2, m_1) = \frac{(1 + \widehat{a}_1)\sigma_1^2}{\sigma_1^2 + \sigma_\varepsilon^2} + \frac{[1 + \widehat{a}_1 + \alpha_2(\widehat{a}_1, \sigma_1^2, \sigma_\varepsilon^2)]\sigma_1^2}{[1 + (1 + \widehat{a}_1)^2]\sigma_1^2 + \sigma_\varepsilon^2} + \frac{(1 + \widehat{a}_1)[1 + \widehat{a}_1 + \alpha_2(\widehat{a}_1, \sigma_1^2, \sigma_\varepsilon^2)]\sigma_1^2 m_1}{[1 + (1 + \widehat{a}_1)^2]\sigma_1^2 + \sigma_\varepsilon^2}$$

is the worker's marginal benefit from effort in period one, while the term

$$MC_{\text{risk}}(a, \widehat{a}_1, \sigma_1^2, \sigma_\varepsilon^2, r) = -r \left\{ \frac{(1 + \widehat{a}_1)[1 + \widehat{a}_1 + \alpha_2(\widehat{a}_1, \sigma_1^2, \sigma_\varepsilon^2)]\sigma_1^2}{[1 + (1 + \widehat{a}_1)^2]\sigma_1^2 + \sigma_\varepsilon^2} \right\}^2 [1 + a + \alpha_2(\widehat{a}_1, \sigma_1^2, \sigma_\varepsilon^2)]\sigma_1^2$$

is the worker's marginal cost of effort a in period one due to his risk aversion. Thus, the possible values for the worker's choice of effort in period one, a_1^* , are the non negative solutions to

$$J_1(a, \lambda, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) = g'(a) - \lambda - \frac{(1 + a)\sigma_1^2}{\sigma_1^2 + \sigma_\varepsilon^2} - \frac{[1 + a + \alpha_2(a, \sigma_1^2, \sigma_\varepsilon^2)]\sigma_1^2}{[1 + (1 + a)^2]\sigma_1^2 + \sigma_\varepsilon^2} - \frac{(1 + a)[1 + a + \alpha_2(a, \sigma_1^2, \sigma_\varepsilon^2)]\sigma_1^2}{[1 + (1 + a)^2]\sigma_1^2 + \sigma_\varepsilon^2} \left\{ m_1 - \frac{r(1 + a)[1 + a + \alpha_2(a, \sigma_1^2, \sigma_\varepsilon^2)]^2 \sigma_1^4}{[1 + (1 + a)^2]\sigma_1^2 + \sigma_\varepsilon^2} \right\} = 0, \quad (20)$$

where $\lambda \geq 0$ and $\lambda = 0$ if a_1^* is positive. By construction, the worker's choice of effort in period two is $\alpha_2(a_1^*, \sigma_1^2, \sigma_\varepsilon^2)$, which is unique given a_1^* .

Proposition 12. *An equilibrium always exists.*

Proof: Suppose first that $J_1(0, 0, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) \geq 0$. Given that

$$J_1(0, \lambda, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) = J_1(0, 0, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) - \lambda,$$

it is immediate to see that $a_1^* = 0$ is a solution to (20) in this case. Suppose now that $J_1(0, 0, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) < 0$. Notice that α_2 uniformly bounded and $\lim_{a \rightarrow \infty} \xi(a) = \infty$ imply that $\lim_{a \rightarrow \infty} J_1(a, 0, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) = \infty$. Since J_1 is continuous in a (as α_2 is continuous in a), the intermediate value theorem implies that (20) has a positive solution. \square

As in the case where $\vartheta_2 = 0$, the worker's choice of effort in period one need not be unique. We now identify necessary and sufficient conditions for the worker's choice of effort in period one to be positive. For this, let

$$j_1(\sigma_1^2, \sigma_\varepsilon^2, m_1, r) = 1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_\varepsilon^2} + [1 + \alpha_2(0, \sigma_1^2, \sigma_\varepsilon^2)](1 + m_1) - \frac{r\sigma_1^4}{2\sigma_1^2 + \sigma_\varepsilon^2} [1 + \alpha_2(0, \sigma_1^2, \sigma_\varepsilon^2)]^3.$$

Note that α_2 uniformly bounded implies that $\lim_{\sigma_1^2 \rightarrow \infty} j_1(\sigma_1^2, \sigma_\varepsilon^2, m_1, r) = -\infty$. Given that $j_1(0, \sigma_\varepsilon^2, m_1, r) = 2 + m_1 > 0$ and j_1 is continuous in σ_1^2 (as α_2 is continuous in σ_1^2), the equation $j_1(\sigma_1^2, \sigma_\varepsilon^2, m_1, r) = 0$ has a solution, and its solutions are positive. Now observe that

$$\begin{aligned} \frac{\partial j_1}{\partial \sigma_1^2}(\sigma_1^2, \sigma_\varepsilon^2, m_1, r) &= \frac{\sigma_\varepsilon^2}{(\sigma_1^2 + \sigma_\varepsilon^2)^2} + (1 + m_1) \frac{\partial \alpha_2}{\partial \sigma_1^2}(0, \sigma_1^2, \sigma_\varepsilon^2) - \frac{2r\sigma_1^2(\sigma_1^2 + \sigma_\varepsilon^2)}{(2\sigma_1^2 + \sigma_\varepsilon^2)^2} [1 + \alpha_2(0, \sigma_1^2, \sigma_\varepsilon^2)]^3 \\ &\quad - \frac{3r\sigma_1^4}{2\sigma_1^2 + \sigma_\varepsilon^2} [1 + \alpha_2(0, \sigma_1^2, \sigma_\varepsilon^2)]^2 \frac{\partial \alpha_2}{\partial \sigma_1^2}(0, \sigma_1^2, \sigma_\varepsilon^2) \\ &< \frac{1}{\sigma_1^2} \underbrace{\left\{ \frac{\sigma_1^2}{\sigma_1^2 + \sigma_\varepsilon^2} - \frac{r\sigma_1^4}{2\sigma_1^2 + \sigma_\varepsilon^2} [1 + \alpha_2(0, \sigma_1^2, \sigma_\varepsilon^2)]^3 \right\}}_A \\ &\quad + \frac{\partial \alpha_2}{\partial \sigma_1^2}(0, \sigma_1^2, \sigma_\varepsilon^2) \underbrace{\left\{ (1 + m_1) - \frac{3r\sigma_1^4}{2\sigma_1^2 + \sigma_\varepsilon^2} [1 + \alpha_2(0, \sigma_1^2, \sigma_\varepsilon^2)]^2 \right\}}_B. \end{aligned}$$

Since $1 + (1 + m_1)[1 + \alpha_2(0, \sigma_1^2, \sigma_\varepsilon^2)] > 0$, the term A is negative when $j_1(\sigma_1^2, \sigma_\varepsilon^2, m_1, r) = 0$. Likewise, $1 + (\sigma_1^2 + \sigma_\varepsilon^2)^{-1}\sigma_1^2 > 0$ implies that B is also negative when $j_1(\sigma_1^2, \sigma_\varepsilon^2, m_1, r) = 0$. Given that $\partial \alpha_2(0, \sigma_1^2, \sigma_\varepsilon^2)/\partial \sigma_1^2 > 0$ by (18), we then have that $\partial j_1(\sigma_1^2, \sigma_\varepsilon^2, m_1, r)/\partial \sigma_1^2 < 0$ when $j_1(\sigma_1^2, \sigma_\varepsilon^2, m_1, r) = 0$. Thus, there exists a unique $\bar{\Sigma}_1^2 = \bar{\Sigma}_1^2(\sigma_\varepsilon^2, m_1, r) > 0$ such that $j_1(\sigma_1^2, \sigma_\varepsilon^2, m_1, r) > 0$ if, and only if, $\sigma_1^2 \in (0, \bar{\Sigma}_1^2)$. We have the following result, which is useful in the analysis that follows.

Lemma 3. *The cutoff $\bar{\Sigma}_1^2$ is strictly increasing in σ_ε^2 , with $\lim_{\sigma_\varepsilon^2 \rightarrow \infty} \bar{\Sigma}_1^2 = \infty$. Moreover, $\lim_{r \rightarrow 0} \bar{\Sigma}_1^2 = \infty$, $\lim_{r \rightarrow \infty} r\bar{\Sigma}_1^2 = \infty$, and*

$$\lim_{\sigma_\varepsilon^2 \rightarrow 0} \bar{\Sigma}_1^2 = \frac{1}{r[1 + a_2^0]^3} \{2 + [1 + a_2^0](1 + m_1)\},$$

where a_2^0 is the unique solution to $g'(a) = (1 + a)/2$.

Proof: Note that

$$\begin{aligned}
\frac{\partial j_1}{\partial \sigma_\varepsilon^2}(\sigma_1^2, \sigma_\varepsilon^2, m_1, r) &= -\frac{\sigma_1^2}{(\sigma_1^2 + \sigma_\varepsilon^2)^2} + (1 + m_1) \frac{\partial \alpha_2}{\partial \sigma_\varepsilon^2}(0, \sigma_1^2, \sigma_\varepsilon^2) + \frac{r\sigma_1^4 [1 + \alpha_2(0, \sigma_1^2, \sigma_\varepsilon^2)]^3}{(2\sigma_1^2 + \sigma_\varepsilon^2)^2} \\
&\quad - 3 [1 + \alpha_2(0, \sigma_1^2, \sigma_\varepsilon^2)]^2 \frac{r\sigma_1^4}{2\sigma_1^2 + \sigma_\varepsilon^2} \frac{\partial \alpha_2}{\partial \sigma_\varepsilon^2}(0, \sigma_1^2, \sigma_\varepsilon^2) \\
&> \frac{1}{2\sigma_1^2 + \sigma_\varepsilon^2} \underbrace{\left\{ \frac{r\sigma_1^4 [1 + \alpha_2(0, \sigma_1^2, \sigma_\varepsilon^2)]^3}{2\sigma_1^2 + \sigma_\varepsilon^2} - 1 - \frac{\sigma_1^2}{\sigma_1^2 + \sigma_\varepsilon^2} \right\}}_C \\
&\quad + \frac{\partial \alpha_2}{\partial \sigma_\varepsilon^2}(0, \sigma_1^2, \sigma_\varepsilon^2) \underbrace{\left\{ 1 + m_1 - \frac{3r\sigma_1^4 [1 + \alpha_2(0, \sigma_1^2, \sigma_\varepsilon^2)]^2}{2\sigma_1^2 + \sigma_\varepsilon^2} \right\}}_D.
\end{aligned}$$

Since both C and D are negative when $\sigma_1^2 = \bar{\Sigma}_1^2$, and α_2 is strictly decreasing in σ_ε^2 , we have that $\partial j_1(\sigma_1^2, \sigma_\varepsilon^2, m_1, r)/\partial \sigma_\varepsilon^2 > 0$ when $\sigma_1^2 = \bar{\Sigma}_1^2$. Hence, by the Implicit Function Theorem, $\bar{\Sigma}_1^2$ is strictly increasing in σ_ε^2 .

Now note that

$$\bar{\Sigma}_1^4 = \frac{2\bar{\Sigma}_1^2 + \sigma_\varepsilon^2}{r[1 + \alpha_2(0, \bar{\Sigma}_1^2, \sigma_\varepsilon^2)]^3} \left\{ 1 + \frac{\bar{\Sigma}_1^2}{\bar{\Sigma}_1^2 + \sigma_\varepsilon^2} + [1 + \alpha_2(0, \bar{\Sigma}_1^2, \sigma_\varepsilon^2)](1 + m_1) \right\}, \quad (21)$$

and so

$$\bar{\Sigma}_1^2 > \frac{\sigma_\varepsilon^2 \left\{ 1 + [1 + \alpha_2(0, \bar{\Sigma}_1^2, \sigma_\varepsilon^2)](1 + m_1) \right\}}{r[1 + \alpha_2(0, \bar{\Sigma}_1^2, \sigma_\varepsilon^2)]^3}.$$

Since α_2 is uniformly bounded, we then have that $\lim_{\sigma_\varepsilon^2 \rightarrow \infty} \bar{\Sigma}_1^2 = \lim_{r \rightarrow 0} \bar{\Sigma}_1^2 = \infty$. Next, note that (21) implies that

$$\bar{\Sigma}_1^4 - \frac{2(1 + m_1)\bar{\Sigma}_1^2}{r[1 + \alpha_2(0, \bar{\Sigma}_1^2, \sigma_\varepsilon^2)]^2} - \frac{\sigma_\varepsilon^2}{r[1 + \alpha_2(0, \bar{\Sigma}_1^2, \sigma_\varepsilon^2)]^3} > 0,$$

from which we obtain that

$$\bar{\Sigma}_1^2 > \frac{1 + m_1}{r[1 + \alpha_2(0, \bar{\Sigma}_1^2, \sigma_\varepsilon^2)]^2} + \left\{ \frac{(1 + m_1)^2}{r^2[1 + \alpha_2(0, \bar{\Sigma}_1^2, \sigma_\varepsilon^2)]^4} + \frac{\sigma_\varepsilon^2}{r[1 + \alpha_2(0, \bar{\Sigma}_1^2, \sigma_\varepsilon^2)]^3} \right\}^{1/2}.$$

Therefore,

$$r\bar{\Sigma}_1^2 > \left\{ \frac{4r\sigma_\varepsilon^2}{[1 + \alpha_2(0, \bar{\Sigma}_1^2, \sigma_\varepsilon^2)]^3} \right\}^{1/2},$$

and so, once more since α_2 is uniformly bounded, $\lim_{r \rightarrow \infty} r\bar{\Sigma}_1^2 = \infty$.

To finish, note from (18) that $\alpha_2(0, \sigma_1^2, 0) \equiv a_2^0$, and so

$$\frac{1}{r[1 + a_2^0]^3} \{2 + [1 + a_2^0](1 + m_1)\}$$

is the only solution to $j_1(\sigma_1^2, 0, m_1, r) = 0$. Given that $\Sigma_1^2 > 0$ implies that $j_1(\cdot, \cdot, m_1, r)$ is continuous at $(\sigma_1^2, \sigma_\varepsilon^2) = (\Sigma_1^2, 0)$, the same argument used in Appendix B to prove that $\lim_{\sigma_\varepsilon^2 \rightarrow 0} \bar{\Sigma}_1^2 = 2(3 + m_1)/r$ when $\vartheta_2 = 0$ can be used to show that

$$\lim_{\sigma_\varepsilon^2 \rightarrow 0} \bar{\Sigma}_1^2 = \frac{1}{r[1 + a_2^0]^3} \{2 + [1 + a_2^0](1 + m_1)\},$$

which completes the proof. \square

The next result, Proposition 13, shows that the worker's effort in period one can only be zero if $\sigma_1^2 \geq \bar{\Sigma}_1^2$, and that this effort is necessarily zero if σ_1^2 is large enough (given σ_ε^2 , m_1 , and r). Moreover, for each m_1 and r , there exists a lower bound on σ_ε^2 above which $\sigma_1^2 \geq \bar{\Sigma}_1^2$ implies that the worker does not exert effort in the first period. Likewise, for each m_1 and σ_ε^2 , there exists a lower bound on r above which $\sigma_1^2 \geq \bar{\Sigma}_1^2$ implies that the worker's effort in period one is zero.

Proposition 13. *The worker's effort in the first period is positive if $\sigma_1^2 \in (0, \bar{\Sigma}_1^2)$. There exists $\tilde{\Sigma}_1^2 = \tilde{\Sigma}_1^2(\sigma_\varepsilon^2, m_1, r) \geq \bar{\Sigma}_1^2$ such that the worker's effort in period one is zero if $\sigma_1^2 \geq \tilde{\Sigma}_1^2$. Moreover, for each m_1 and r , there exists $\Sigma_\varepsilon^2 \geq 0$ such that $\tilde{\Sigma}_1^2 = \bar{\Sigma}_1^2$ if $\sigma_\varepsilon^2 > \Sigma_\varepsilon^2$, and for each m_1 and σ_ε^2 , there exists $\bar{r} \geq 0$ such that $\tilde{\Sigma}_1^2 = \bar{\Sigma}_1^2$ if $r > \bar{r}$.*

Proof: First note that $J_1(0, \lambda, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) = 0$ if, and only if,

$$\lambda = -\frac{\sigma_1^2}{2\sigma_1^2 + \sigma_\varepsilon^2} j_1(\sigma_1^2, \sigma_\varepsilon^2, m_1, r),$$

and that $\lambda \geq 0$ if, and only if, $\sigma_1^2 \geq \bar{\Sigma}_1^2$. Hence, $a_1^* = 0$ is a solution to (20) only when $\sigma_1^2 \geq \bar{\Sigma}_1^2$. Since, by Proposition 12, a solution to (20) always exists, we then have that the solutions to (20) are positive when $\sigma_1^2 \in (0, \bar{\Sigma}_1^2)$.

We now show that there exists $\tilde{\Sigma}_1^2 = \tilde{\Sigma}_1^2(\sigma_\varepsilon^2, m_1, r) \geq \bar{\Sigma}_1^2$ such that $a_1^* = 0$ is the only solution to (20) when $\sigma_1^2 \geq \tilde{\Sigma}_1^2$. For this, let $\mathcal{J}_1(a, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) = (1 + a)^{-1} J_1(a, 0, \sigma_1^2, \sigma_\varepsilon^2, m_1, r)$. Note that (20) has no positive solution if $\mathcal{J}_1(a, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) > 0$ for all $a > 0$. Hence, we are done with this part of the argument if we show that there exists $\tilde{\Sigma}_1^2 \geq \bar{\Sigma}_1^2$ such that

$\mathcal{J}_1(a, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) > 0$ for all $a > 0$ when $\sigma_1^2 \geq \widetilde{\Sigma}_1^2$. For ease of notation, we omit the dependence of α_2 on σ_1^2 and σ_ε^2 in the remainder of the proof.

To start, let \widetilde{a}_1 be the value of a such that $(1+a)^2\sigma_1^2 = \sigma_1^2 + \sigma_\varepsilon^2$. We claim that $\sigma_1^2 \geq \widetilde{\Sigma}_1^2$ implies that $\mathcal{J}_1(a, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) > 0$ for all $a \in (0, \widetilde{a}_1)$. First observe from (19) that

$$1 + \frac{\partial \alpha_2}{\partial \widehat{a}_1}(a) \geq g''(\alpha_2(a)) \left\{ g''(\alpha_2(a)) - \frac{(1+a)\sigma_1^2}{[1+(1+a)^2]\sigma_1^2 + \sigma_\varepsilon^2} \right\}^{-1}$$

if $a \in (0, \widetilde{a}_1)$. Since $g''(a) \geq g'(a)/a$ for all $a > 0$, we also have that

$$g''(\alpha_2(a)) > \frac{(1+a)^2\sigma_1^2}{\alpha_2(a) \{[1+(1+a)^2]\sigma_1^2 + \sigma_\varepsilon^2\}} + \frac{(1+a)\sigma_1^2}{[1+(1+a)^2]\sigma_1^2 + \sigma_\varepsilon^2} \quad (22)$$

for all $a > 0$. Therefore,

$$1 + \frac{\partial \alpha_2}{\partial \widehat{a}_1}(a) > \frac{1+a+\alpha_2(a)}{1+a}$$

if $a \in (0, \widetilde{a}_1)$. In particular, $a \in (0, \widetilde{a}_1)$ implies that $(1+a)^{-1}[1+a+\alpha_2(a)]^2$ is strictly increasing in a and

$$2(1+a)[1+a+\alpha_2(a)]^2 \left[1 + \frac{\partial \alpha_2}{\partial \widehat{a}_1}(a) \right] \geq 2[1+a+\alpha_2(a)]^3.$$

Now note (omitting the algebra) that

$$\begin{aligned} \frac{\partial \mathcal{J}_1}{\partial a}(a, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) &= \xi'(a) + \frac{[1+a+\alpha_2(a)]\sigma_1^2}{(1+a)^2\{[1+(1+a)^2]\sigma_1^2 + \sigma_\varepsilon^2\}} + \frac{2[1+a+\alpha_2(a)]\sigma_1^4}{\{[1+(1+a)^2]\sigma_1^2 + \sigma_\varepsilon^2\}^2} \\ &\quad + \frac{2(1+a)[1+a+\alpha_2(a)]\sigma_1^4 m_1}{[1+(1+a)^2]\sigma_1^2 + \sigma_\varepsilon^2} - \left[1 + \frac{\partial \alpha_2(a)}{\partial \widehat{a}_1} \right] \frac{A\sigma_1^2}{[1+(1+a)^2]\sigma_1^2 + \sigma_\varepsilon^2} \\ &\quad + \frac{B[1+a+\alpha_2(a)]r\sigma_1^6}{\{[1+(1+a)^2]\sigma_1^2 + \sigma_\varepsilon^2\}^2} - \frac{4r(1+a)^2[1+a+\alpha_2(a)]^3\sigma_1^8}{\{[1+(1+a)^2]\sigma_1^2 + \sigma_\varepsilon^2\}^3}, \end{aligned}$$

where

$$A = \frac{1}{1+a} + m_1 + \frac{r\sigma_1^4(1+a)[1+a+\alpha_2(a)]^2}{[1+(1+a)^2]\sigma_1^2 + \sigma_\varepsilon^2} \quad \text{and} \quad B = 1+a+\alpha_2(a) + 2(1+a) \left[1 + \frac{\partial \alpha_2}{\partial \widehat{a}_1}(a) \right].$$

From the previous paragraph, we have that: (i)

$$\begin{aligned} A &< 1 + m_1 - \frac{r(1+a)^2\sigma_1^4}{[1+(1+a)^2]\sigma_1^2 + \sigma_\varepsilon^2} \frac{[1+a+\alpha_2(a)]^2}{1+a} < 1 + m_1 - \frac{[1+\alpha_2(0)]^2 r \sigma_1^4}{2\sigma_1^2 + \sigma_\varepsilon^2} \\ &= \frac{1}{1+\alpha_2(0)} \left[j_1(\sigma_1^2, \sigma_\varepsilon^2, m_1, r) - 1 - \frac{\sigma_1^2}{\sigma_1^2 + \sigma_\varepsilon^2} \right] < 0 \end{aligned}$$

if $a \in (0, \tilde{a}_1)$ and $\sigma_1^2 \geq \bar{\Sigma}_1^2$; and (ii) $B \geq 3[1 + a + \alpha_2(a)]$ if $a \in (0, \tilde{a}_1)$. Given that $a \in (0, \tilde{a}_1)$ also implies that

$$\frac{4r(1+a)^2[1+a+\alpha_2(a)]^3\sigma_1^8}{\{[1+(1+a)^2]\sigma_1^2+\sigma_\varepsilon^2\}^3} \leq \frac{2r[1+a+\alpha_2(a)]^3\sigma_1^6}{\{[1+(1+a)^2]\sigma_1^2+\sigma_\varepsilon^2\}^2},$$

we can then conclude that $\partial\mathcal{J}_1(a, \sigma_1^2, \sigma_\varepsilon^2, m_1, r)/\partial a > 0$ for all $a \in (0, \tilde{a}_1)$ when $\sigma_1^2 \geq \bar{\Sigma}_1^2$. Since $\mathcal{J}_1(0, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) \geq 0$ if $\sigma_1^2 \geq \bar{\Sigma}_1^2$, we then have that $\mathcal{J}_1(a, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) > 0$ for all $a \in (0, \tilde{a}_1)$ when $\sigma_1^2 \geq \bar{\Sigma}_1^2$, as claimed.

We now establish that for each σ_ε^2 , m_1 , and r , there exists $\tilde{\Sigma}_1^2 \geq \bar{\Sigma}_1^2$ with the property that $\mathcal{J}_1(a, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) > 0$ for all $a \geq \tilde{a}_1$ if $\sigma_1^2 \geq \tilde{\Sigma}_1^2$. For this, let

$$\begin{aligned} G(a, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) &= \xi(a) - \frac{\sigma_1^2}{\sigma_1^2 + \sigma_\varepsilon^2} - \frac{[1+a+\alpha_2(a)]\sigma_1^2}{(1+a)\{[1+(1+a)^2]\sigma_1^2+\sigma_\varepsilon^2\}} \\ &\quad - \frac{[1+a+\alpha_2(a)]\sigma_1^2 m_1}{[1+(1+a)^2]\sigma_1^2+\sigma_\varepsilon^2} + \frac{r(1+a)^4\sigma_1^6}{\{[1+(1+a)^2]\sigma_1^2+\sigma_\varepsilon^2\}^2}. \end{aligned}$$

By construction, $G(a, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) \leq \mathcal{J}_1(a, \sigma_1^2, \sigma_\varepsilon^2, m_1, r)$ for all $a \geq 0$. We claim that G is strictly increasing in a if $a \geq \tilde{a}_1$. Since $(1+a)\sigma_1^2/\{[1+(1+a)^2]\sigma_1^2+\sigma_\varepsilon^2\}$ is strictly decreasing in a when $a \geq \tilde{a}_1$, and

$$\frac{[1+a+\alpha_2(a)]\sigma_1^2 m_1}{[1+(1+a)^2]\sigma_1^2+\sigma_\varepsilon^2} = \frac{1+a}{[1+(1+a)^2]\sigma_1^2+\sigma_\varepsilon^2} \frac{1+a+\alpha_2(a)}{1+a},$$

the desired result holds as long as $\zeta(a) = (1+a)^{-1}[1+a+\alpha_2(a)]$ is decreasing in a when $a \geq \tilde{a}_1$. Now observe from (19) and (22) that

$$\frac{\partial\alpha_2}{\partial\hat{a}_1}(a) \leq \frac{\alpha_2(a)}{(1+a)^2} \left\{ 2(1+a) + \alpha_2(a) - \frac{2(1+a)^2[1+a+\alpha_2(a)]\sigma_1^2}{[1+(1+a)^2]\sigma_1^2+\sigma_\varepsilon^2} \right\},$$

and so $\partial\alpha_2(a)/\partial\hat{a}_1 \leq (1+a)^{-1}\alpha_2(a)$ if $a \geq \tilde{a}_1$. Given that ζ is decreasing in a if, and only if, $(1+a)\partial\alpha_2(a)/\partial\hat{a}_1 \leq \alpha_2$, the function G is indeed strictly increasing in a when $a \geq \tilde{a}_1$. Consequently, $\mathcal{J}_1(a, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) > 0$ for all $a \geq \tilde{a}_1$ as long as $G(\tilde{a}_1, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) \geq 0$. Since $\lim_{\sigma_1^2 \rightarrow \infty} G(\tilde{a}_1, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) = \infty$, there exists $\tilde{\Sigma}_1^2 \in [\bar{\Sigma}_1^2, \infty)$ such that $G(\tilde{a}_1, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) \geq 0$ for all $\sigma_1^2 \geq \tilde{\Sigma}_1^2$. By construction, $\sigma_1^2 \geq \tilde{\Sigma}_1^2$ implies that $\mathcal{J}_1(a, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) > 0$ for all $a \geq \tilde{a}_1$, the desired result.

To finish the proof, we show that for each m_1 and r , there exists $\Sigma_\varepsilon^2 \geq 0$ such that $G(\tilde{a}_1, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) \geq 0$ for all $\sigma_1^2 \geq \bar{\Sigma}_1^2$ when $\sigma_\varepsilon^2 > \Sigma_\varepsilon^2$, and that for each m_1 and σ_ε^2 , there

exists $\bar{r} \geq 0$ such that $G(\tilde{a}_1, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) \geq 0$ for all $\sigma_1^2 \geq \bar{\Sigma}_1^2$ when $r > \bar{r}$. First note, since α_2 is uniformly bounded by \bar{a}_2 , that $\sigma_1^2 \geq \bar{\Sigma}_1^2$ implies that

$$\begin{aligned} G(\tilde{a}_1, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) &= \xi(\tilde{a}_1) - \frac{\sigma_1^2}{\sigma_1^2 + \sigma_\varepsilon^2} - \frac{g'(\alpha_2(\tilde{a}_1))}{1 + \tilde{a}_1} \left[\frac{1}{1 + \tilde{a}_1} + m_1 \right] + \frac{r\sigma_1^2}{4} \\ &> \xi \left(\sqrt{1 + \frac{\sigma_\varepsilon^2}{\bar{\Sigma}_1^2}} - 1 \right) - 1 - (1 + m_1)g'(\bar{a}_2) + \frac{r\bar{\Sigma}_1^2}{4} = \mathcal{G}(\sigma_\varepsilon^2, m_1, r). \end{aligned}$$

Now let $\Sigma_\varepsilon^2 = \Sigma_\varepsilon^2(m_1, r) = \inf\{\sigma_\varepsilon^2 > 0 : \mathcal{G}(\sigma_\varepsilon^2, m_1, r) \geq 0\}$. Since $\lim_{\sigma_\varepsilon^2 \rightarrow \infty} \bar{\Sigma}_1^2 = \infty$ by Lemma 3, we have that $\lim_{\sigma_\varepsilon^2 \rightarrow \infty} \mathcal{G}(\sigma_\varepsilon^2, m_1, r) = \infty$, which implies that $\Sigma_\varepsilon^2 < \infty$. Likewise, let $\bar{r} = \bar{r}(m_1, \sigma_\varepsilon^2) = \inf\{r > 0 : \mathcal{G}(\sigma_\varepsilon^2, m_1, r) \geq 0\}$. Since $\lim_{r \rightarrow \infty} r\bar{\Sigma}_1^2 = \infty$ by Lemma 3, we also have that $\lim_{r \rightarrow \infty} \mathcal{G}(\sigma_\varepsilon^2, m_1, r) = \infty$, which implies that $\bar{r} < \infty$ as well. By construction, $G(\tilde{a}_1, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) \geq 0$ for all $\sigma_1^2 \geq \bar{\Sigma}_1^2$ when either $\sigma_\varepsilon^2 > \Sigma_\varepsilon^2$ or $r > \bar{r}$, in which case $\tilde{\Sigma}_1^2 = \bar{\Sigma}_1^2$. This completes the proof. \square

As in Proposition 6 in the main text, Proposition 13 shows that for each σ_ε^2 , m_1 , and r , there exists a cutoff $\bar{\Sigma}_1^2 = \bar{\Sigma}_1^2(\sigma_\varepsilon^2, m_1, r)$ such that the worker's effort in period one is positive if $\sigma_1^2 \in (0, \bar{\Sigma}_1^2)$. Moreover, for each σ_ε^2 , m_1 , and r , there exists $\tilde{\Sigma}_1^2 = \tilde{\Sigma}_1^2(\sigma_\varepsilon^2, m_1, r) \geq \bar{\Sigma}_1^2$ such that the worker's effort in period one is zero if $\sigma_1^2 \geq \tilde{\Sigma}_1^2$, where $\tilde{\Sigma}_1^2 = \bar{\Sigma}_1^2$ if either the noise in output is high enough or the worker is sufficiently risk averse. The next result shows that there exist cost functions g for which $\tilde{\Sigma}_1^2 = \bar{\Sigma}_1^2$ regardless of σ_ε^2 , m_1 , and r .

Corollary 1. *For a non-empty set of cost functions g , effort in the first period is positive if $\sigma_1^2 \in (0, \bar{\Sigma}_1^2)$ and zero otherwise.*

Proof: In what follows, we omit the dependence of α_2 on σ_1^2 and σ_ε^2 when convenient. We divide the argument in several steps. First we show (Step 1) that $g'(1/5) > 6/5$ implies that

$$\alpha_2(a) \leq (1 + a)/5 \tag{23}$$

for all $a \geq 0$. Indeed, by (18), condition (23) is satisfied if

$$g' \left(\frac{1 + a}{5} \right) \geq \frac{6}{5} \frac{(1 + a)^2 \sigma_1^2}{[1 + (1 + a)^2] \sigma_1^2 + \sigma_\varepsilon^2}.$$

The desired result holds since the right side of the above inequality is bounded above by $6/5$ and g' is nondecreasing. In what follows we assume that $g'(1/5) > 6/5$.

Let \tilde{a}_1 be the value of a such that $(1+a)^2\sigma_1^2 = 2(\sigma_1^2 + \sigma_\varepsilon^2)$. Now we show (Step 2) that α_2 is strictly increasing in \hat{a}_1 if $\hat{a}_1 \in (0, \tilde{a}_1)$. Recall from (19) that $\partial\alpha_2(a)/\partial\hat{a}_1 > 0$ if, and only if,

$$2(1+a) \left\{ 1 - \frac{(1+a)^2\sigma_1^2}{[1+(1+a)^2]\sigma_1^2 + \sigma_\varepsilon^2} \right\} + \alpha_2(a) \left\{ 1 - \frac{2(1+a)^2\sigma_1^2}{[1+(1+a)^2]\sigma_1^2 + \sigma_\varepsilon^2} \right\} \\ \propto 2(1+a)(\sigma_1^2 + \sigma_\varepsilon^2) + \alpha_2(a) [\sigma_1^2 + \sigma_\varepsilon^2 - (1+a)^2\sigma_1^2] > 0.$$

We know from the proof of Proposition 13 that $\partial\alpha_2(a)/\partial\hat{a}_1 > 0$ if $\hat{a}_1 \in (0, \tilde{a}_1]$. So, assume that $a \in [\tilde{a}_1, \tilde{a}_1)$, in which case $\partial\alpha_2(a)/\partial\hat{a}_1 > 0$ if, and only if,

$$\alpha_2(a) \leq \frac{2(1+a)(\sigma_1^2 + \sigma_\varepsilon^2)}{(1+a)^2\sigma_1^2 - (\sigma_1^2 + \sigma_\varepsilon^2)}.$$

Since $\alpha_2(a) \leq (1+a)$ by (23), a sufficient condition for the last inequality is that $(1+a)^2\sigma_1^2 \leq 3(\sigma_1^2 + \sigma_\varepsilon^2)$, which is satisfied by assumption.

The next step (Step 3) consists in showing that $\mathcal{J}_1(a, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) > 0$ for all $a \in (0, \tilde{a}_1)$ if $\sigma_1^2 \geq \bar{\Sigma}_1^2$. First note that $(1+a)^{-1}[1+a+\alpha_2(a)]^2$ is strictly increasing in a when $a \in (0, \tilde{a}_1)$ if, and only if,

$$2(1+a) \left[1 + \frac{\partial\alpha_2}{\partial\hat{a}_1}(a) \right] - [1+a+\alpha_2(a)] > 0$$

for all $a \in (0, \tilde{a}_1)$, which holds by (23) and Step 2. Now observe that $a \in (0, \tilde{a}_1)$ implies that

$$\frac{4r(1+a)^2[1+a+\alpha_2(a)]^3\sigma_1^8}{\{[1+(1+a)^2]\sigma_1^2 + \sigma_\varepsilon^2\}^3} \leq \frac{8}{3} \frac{r[1+a+\alpha_2(a)]^3\sigma_1^6}{\{[1+(1+a)^2]\sigma_1^2 + \sigma_\varepsilon^2\}^2}$$

and that (23) and Step 2 imply that

$$[1+a+\alpha_2(a)]^3 + 2(1+a)[1+a+\alpha_2(a)]^2 \left[1 + \frac{\partial\alpha_2}{\partial\hat{a}_1}(a) \right] \geq \frac{8}{3}[1+a+\alpha_2(a)]^3$$

when $a \in (0, \tilde{a}_1)$. Hence, by the proof of Proposition 13, $\partial\mathcal{J}_1(a, \sigma_1^2, \sigma_\varepsilon^2, m_1, r)/\partial a > 0$ for all $a \in (0, \tilde{a}_1)$ when $\sigma_1^2 \geq \bar{\Sigma}_1^2$, which implies the desired result.

Let $G(a, \sigma_1^2, \sigma_\varepsilon^2, m_1, r)$ be the same as in the proof of Proposition 13. From Step 3 and the proof of Proposition 13, we have that $\mathcal{J}_1(a, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) > 0$ for all $a > 0$ when $\sigma_1^2 \geq \bar{\Sigma}_1^2$ as long as

$$G(\tilde{a}_1, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) = \xi(\tilde{a}_1) - \frac{\sigma_1^2}{\sigma_1^2 + \sigma_\varepsilon^2} - \frac{g'(\alpha_2(\tilde{a}_1))}{1 + \tilde{a}_1} \left[\frac{1}{1 + \tilde{a}_1} + m_1 \right] + \frac{4r\sigma_1^2}{9} \geq 0$$

for all $\sigma_1^2 \geq \bar{\Sigma}_1^2$. Observe that (23) implies that

$$g'(\alpha_2(a)) \leq \frac{6}{5} \frac{(1+a)^2 \sigma_1^2}{[1 + (1+a)^2] \sigma_1^2 + \sigma_\varepsilon^2},$$

and so $g'(\alpha_2(\tilde{a}_1, \sigma_1^2, \sigma_\varepsilon^2)) \leq 4/5$. Moreover, $1 + \tilde{a}_1 \geq \sqrt{2}$. Thus,

$$G(\tilde{a}_1, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) \geq \xi(\sqrt{2}) - 1 - \frac{4}{5\sqrt{2}}(1 + m_1) + \frac{4r\sigma_1^2}{9}.$$

From the definition of $\bar{\Sigma}_1^2$, we have that

$$\frac{4r\bar{\Sigma}_1^2}{9} \geq \frac{8(1 + m_1)}{9[1 + \alpha_2(0, \bar{\Sigma}_1^2, \sigma_\varepsilon^2)]^2} \geq \frac{50(1 + m_1)}{81},$$

where the second inequality follows from (23). Therefore,

$$G(\tilde{a}_1, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) \geq \xi(\sqrt{2}) - 1 + \underbrace{\left[\frac{50}{81} - \frac{4}{5\sqrt{2}} \right]}_{>0} (1 + m_1),$$

and so $G(\tilde{a}_1, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) \geq 0$ for all $\sigma_1^2 \geq \bar{\Sigma}_1^2$ as long as $\xi(\sqrt{2}) \geq 1$. We can then conclude that if g is such that $g'(1/5) > 6/5$ and $g'(\sqrt{2}) > 1 + \sqrt{2}$, then $\sigma_1^2 \geq \bar{\Sigma}_1^2$ implies that the worker's effort in period one is zero. \square

It is immediate to see from the proof of Corollary 1 that the set of cost functions for which effort in period one is positive if, and only if, $\sigma_1^2 \in (0, \bar{\Sigma}_1^2)$ is robust to small perturbations.

Comparative Statics

For each $\chi = (\sigma_1^2, \sigma_\varepsilon^2, m_1, r) \in \mathbb{R}_{++}^4$, let

$$A_1(\chi) = \{a \in \mathbb{R}_+ : \exists \lambda \geq 0 \text{ with } J_1(a, \lambda, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) = 0 \text{ and } \lambda a = 0\}$$

and define $\mathcal{A}_1 : \mathbb{R}_{++}^4 \rightrightarrows \mathbb{R}_+$ to be such that $\mathcal{A}_1(\chi) = A_1(\chi)$. As in the main text, \mathcal{A}_1 is the correspondence that maps the set of parameters of the model into the set of possible period one effort choices by the worker. Since

$$J_1(a, \lambda, \sigma_1^2, \sigma_\varepsilon^2, m_1, r) \geq g'(a) - (1 + a) - (1 + a + \bar{a}_2)(1 + m_1),$$

for each $\chi \in \mathbb{R}_{++}^4$, the elements of $\mathcal{A}_1(\chi)$ are bounded above by $\bar{a}_1 = \bar{a}_1(m_1)$, where \bar{a}_1 is the unique solution to $g'(a) = (1 + a) + (1 + a + \bar{a}_2)(1 + m_1)$.

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